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Double Complex Bäcklund Transformations of the Axisymmetric Stationary and Cylindrical Symmetric Einstein Vacuum Field and a Geroch Group Structure

Hai-Yang Zhong[1] and Zai-Zhe Zhong[1]

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Neugebauer's results are doublized and the double complex transformations of the axisymmetric stationary vacuum field and the cylindrically symmetric vacuum field are obtained. Neugebauer's original results correspond to half of ours. By making use of these double Bäcklund transformations, we can construct a kind of double realization of the Geroch group. This cannot be achieved from Neugebauer's original results. Moreover, it is more general than Zhong's.

1. INTRODUCTION

We have discussed the axisymmetric stationary Einstein vacuum field (ASVF) by the double complex method, and have applied the method in many aspects (Zhong, 1985, 1988, 1989, 1990*a*,*b*). By this method we have also constructed a kind of double complex realization of the Geroch group (Geroch, 1971, 1972; Zhong, 1990*b*) and made the geometrical construction extremely clear. On the other hand, Neugebauer (1979) has developed a method for working out Bäcklund transformations, which is of effect in studying the Ernst equation (Ernst, 1967, 1968). Naturally, we hope it can be doublized, and a double Bäcklund transformation similar to Neugebauer's will be developed. Furthermore, by using these double Bäcklund transformations we develop a realization of the Geroch group. That is the aim of the present paper. From our results we can see that Neugebauer's results correspond to only half of ours, and from his Bäcklund transformations the construction of the Geroch group cannot be obtained. In addition, the original NK method cannot be applied to the case of the

[1]Department of Physics, Liaoning Normal University, Dalian, 116022, Liaoning, People's Republic of China.

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cylindrically symmetric vacuum field (CSVF). We shall see that our double complex method can also be applied to cylindrical gravitational solutions in the same way. Although half of our hyperbolic solutions cannot represent real physical solutions, these solutions reveal the symmetric structure in the cylindrical gravitational wave solution set.

First, for convenience, we introduce some necessary symbols and draw some relevant conclusions directly. Let J denote the double imaginary unit, i.e., $J = i$ ($i^2 = -1$) or $J = \varepsilon$ ($\varepsilon^2 = +1, \varepsilon \neq \pm 1$). If $a = \{a_0, a_1 \ldots a_n \ldots\}$ is a real sequence and $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $a(J) = \sum a_n J^{2n}$ is called a double real number which corresponds to a real number pair (a_C, a_H) $(a_{(J=i)}, a_{(J=\varepsilon)})$. If both $a(J)$ and $b(J)$ are double real numbers, then $Z(J) = a(J) + Jb(J)$ is called a double complex number which corresponds to the complex number pair $(Z_C, Z_H) \equiv (a_C + ib_C, a_H + \varepsilon b_H)$.

The equation

$$\mathrm{Re}(\mathscr{E}(J))\, \nabla^2 \mathscr{E}(J) = \nabla \mathscr{E}(J)\, \nabla \mathscr{E}(J) \tag{1}$$

$$\nabla^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2$$

$$\nabla = (\partial_\rho, \partial_z)$$

is called the double complex Ernst equation (Zhong, 1985). If $\mathscr{E}(J) = F(J) + J \cdot \Omega(J)$ is a solution of equation (1) and the line element of ASVF is taken as

$$ds^2 = f(dt - \omega\, d\theta)^2 - f^{-1}[e^\Gamma(d\rho^2 + dz^2) + \rho^2\, d\theta^2] \tag{2}$$

where (ρ, z, θ) are the cylindrical coordinates, f and ω are functions of ρ and z only, and the function Γ is determined by the functions f and ω, then corresponding to this line element, we can obtain the physical (real) solution pairs (f, ω), $(\hat{f}, \hat{\omega})$ and the nonphysical (imaginary) solution pairs (f', ω'), $(\hat{f}', \hat{\omega}')$ as follows:

$$\begin{aligned} (f, \omega) &= (F_c, V_{F_c}^{-1}(\Omega_c)) \\ (\hat{f}, \hat{\omega}) &= (T(F_H), \Omega_H) \\ (f', \omega') &= (T(F_c), i\Omega_c) \\ (\hat{f}', \hat{\omega}') &= (F_H, iV_{F_H}^{-1}(\Omega_H)) \end{aligned} \tag{3}$$

where (T, V) is the NK transformation

$$\begin{aligned} T:\ & f \to T(f) = \rho/f, \qquad V_f:\ \omega \to \varphi = V_f(\omega) \\ & \partial_\rho \varphi = \rho^{-1} f^2 \partial_z \omega, \qquad \partial_z \varphi = -\rho^{-1} f^2 \partial_\rho \omega \end{aligned} \tag{4}$$

In the case of CSVF, the line element is taken as

$$ds^2 = \Lambda(dt^2 - dz^2) - g^{-1} t[(dx - \mu\, dy)^2 + g^2\, dy^2] \tag{5}$$

where g and μ are functions of t and z only. Λ is determined by g and μ. Let us consider another double complex Ernst equation,

$$\begin{gathered}\mathrm{Re}(\mathscr{C}(J))\,\tilde{\nabla}^2\mathscr{C}(J)=\tilde{\nabla}\mathscr{C}(J)\cdot\tilde{\nabla}\mathscr{C}(J)\\ \tilde{\nabla}^2=\partial_t^2+t^{-1}\partial_t-\partial_z^2,\qquad \tilde{\nabla}=(\partial_t, i\partial_z)\end{gathered}\tag{6}$$

If $\mathscr{C}(J)=G(J)+JM(J)$ is a known solution, then we can obtain the real solutions (g,μ) and $(\hat{g},\hat{\mu})$, and pure imaginary solutions (g',μ') and $(\hat{g}',\hat{\mu}')$, of CSVF as follows:

$$\begin{aligned}(g,\mu)&=(G_c,M_c)\\ (\hat{g},\hat{\mu})&=(T(G_c),W_{G_c}(M_c))\\ (g',\mu')&=(G_H,iM_H)\\ (\hat{g}',\hat{\mu}')&=(T(G_H),iW_{T(GH)}(M_H)),\end{aligned}\tag{7}$$

where (T, W) is another NK transformation different from (4) (Zhong, 1990*b*):

$$\begin{aligned}T\colon\quad g\to T(g)=t/g,\qquad W_g\colon\quad \mu\to\psi=W_g(\mu)\\ \partial_t\psi=g^{-2}t\partial_z\mu,\qquad \partial_z\psi=g^{-2}t\partial_t\mu\end{aligned}\tag{8}$$

2. DOUBLE BÄCKLUND TRANSFORMATIONS OF ASVF

First we discuss how Neugebauer's method can be doublized. We consider the double complex equations

$$\begin{aligned}\frac{1+J^2}{2}M_{i,1}+\frac{1-J^2}{2}M_{i,2}&=\frac{1+J^2}{2}(D_i^{kl}M_kM_l+E_i^{kl}M_kN_l)+\frac{1-J^2}{2}C_i^{kl}M_kN_l\\ \frac{1+J^2}{2}N_{i,2}+\frac{1-J^2}{2}N_{i,1}&=\frac{1+J^2}{2}(D_i^{kl}N_kN_l+E_i^{kl}N_kM_l)+\frac{1-J^2}{2}C_i^{kl}N_kM_l\\ &(i=1,2,3)\end{aligned}\tag{9}$$

where $M_i(x^1,x^2;J)$ and $N_i(x^1,x^2,J)$ are the double complex functions of x^1 and x^2, $x^1=\rho+Jz$, $x^2=\rho-Jz$, and C_i^{kl}, D_i^{kl}, and E_i^{kl} are all real coefficients. If one takes the functions $M_i(J)$ and $N_i(J)$ as

$$\begin{aligned}M_1(J)&=\frac{\partial_1\mathscr{E}(J)}{2F(J)},\qquad M_2(J)=\frac{\partial_1\bar{\mathscr{E}}(J)}{2F(J)},\qquad M_3=\frac{1}{2\rho}\\ N_1(J)&=\frac{\partial_2\bar{\mathscr{E}}(J)}{2F(J)},\qquad N_2(J)=\frac{\partial_2\mathscr{E}(J)}{2F(J)},\qquad N_3=\frac{1}{2\rho}\end{aligned}\tag{10}$$

and nonvanishing coefficients as

$$\begin{aligned} &E_1^{32} = E_1^{13} = E_2^{31} = E_2^{23} = -\frac{1}{2}, \qquad E_3^{33} = 1 \\ &D_1^{11} = -D_1^{12} = D_2^{22} = -D_2^{21} = 1 \\ &C_1^{11} = C_2^{22} = C_3^{33} = -C_1^{12} = -C_2^{21} = -1 \\ &C_1^{32} = C_1^{13} = C_2^{31} = C_2^{23} = C_2^{23} = -\frac{1}{2} \end{aligned} \tag{11}$$

where $\bar{\mathscr{E}}$ denotes the double complex conjugation of $\mathscr{E}$, then it can be proved directly that (9) is just (1), i.e., the double complex Ernst equation. Therefore, according to a method similar to Neugebauer's, from a known solution $\mathring{M}_i(J)$ and $\mathring{N}_i(J)$ of (9) we can obtain new double solutions $M_i(J)$ and $N_i(J)$ of (9) by the following operation I:

$$I: \quad M_i(J) = \alpha_i^k(J)\mathring{M}_k(J), \qquad N_i(J) = \alpha_i^{-1k}(J)\mathring{N}_k(J) \tag{12}$$

$$(\alpha_i^k(J)) = \begin{pmatrix} \alpha(J) & & \\ & \beta(J) & \\ & & \gamma(J) \end{pmatrix}, \qquad \alpha\beta = \gamma$$

where α_i^{-1k} is the inverse of α_i^k, and α and γ are solutions of the following double complex Riccati equations:

$$d\gamma = (\gamma^2 - \gamma)\mathring{M}_3(J)\, dx^1 + (\gamma - 1)\mathring{N}_3\, dx^2 \tag{13}$$

$$\begin{aligned} d(\alpha\gamma^{-1/2}) = {} & [-\gamma^{1/2}M_2 + (\alpha\gamma^{-1/2})(\mathring{M}_2 - \mathring{M}_1) + (\alpha\gamma^{-1/2})^2\mathring{M}_1\gamma^{1/2}]\, dx^1 \\ & + [-\gamma^{-1/2}\mathring{N}_1 + (\alpha\gamma^{-1/2})(\mathring{N}_1 - \mathring{N}_2) + (\alpha\gamma^{-1/2})^2\mathring{N}_2\gamma^{-1/2}]\, dx^2 \end{aligned} \tag{14}$$

The general solution of equation (13) is easy to find and is given by, when $J = i$,

$$\gamma_c = \frac{k - ix^2}{k + ix^1} \tag{15}$$

and when $J = \varepsilon$,

$$\gamma_H = \frac{k - \varepsilon x^1}{k + \varepsilon x^2} \tag{16}$$

where k is an integral constant. To find the solutions of equation 14 is somewhat complicated; Kyriakopoulos (1988) has managed to obtain one.

As we can see, when $J = i$, the new solutions $M_i(i)$ and $N_i(i)$ obtained by the operation I are exactly what Neugebauer presented. But in the result another half is lost, for Neugebauer's equation only gave half of (9). The lost part is just the case of $J = \varepsilon$.

Consequently, we can present the explicit expressions of the Bäcklund transformation of ASVF as follows:

$$\begin{aligned}\partial_1 \mathscr{E}(J) &= \alpha \frac{\mathscr{E}(J) + \bar{\mathscr{E}}(J)}{\mathring{\mathscr{E}}(J) + \bar{\mathring{\mathscr{E}}}(J)} \partial_1 \mathring{\mathscr{E}}(J) \\ \partial_2 \mathscr{E}(J) &= \frac{\gamma}{\alpha} \frac{\mathscr{E}(J) + \bar{\mathscr{E}}(J)}{\mathring{\mathscr{E}}(J) + \bar{\mathring{\mathscr{E}}}(J)} \partial_2 \mathring{\mathscr{E}}(J)\end{aligned} \tag{17}$$

where $\mathring{\mathscr{E}}(J)$ is the known solution of the double complex Ernst equation and $\alpha(J)$ and $\gamma(J)$ are the solutions of equations (13) and (14).

It is easy to verify that the transformation I possesses the group property, i.e.,

$$\begin{aligned}I(\alpha_2, \gamma_2) I(\alpha_1, \gamma_1) &= I(\alpha_1 \alpha_2, \gamma_1 \gamma_2) \\ I^{-1}(\alpha, \gamma) &= I(\alpha^{-1}, \gamma^{-1})\end{aligned} \tag{18}$$

If α, β, and γ are taken as

$$\alpha = -\frac{\bar{\mathscr{E}}(J) - J \cdot m(J)}{\mathscr{E}(J) + J \cdot m(J)}, \qquad \beta = \frac{1}{\alpha}, \qquad \gamma = 1 \tag{19}$$

and are taken back into (12), in this special case we see, after integration, that the result is the famous double Ehlers transformation

$$\mathscr{E}'(J) = \frac{a(J)\mathscr{E}(J) + Jb(J)}{Jc(J)\mathscr{E}(J) + d(J)} \tag{20}$$

where the double complex real constants $a(J)$, $b(J)$, $c(J)$, and $d(J)$ satisfy the condition $ad - J^2 bc = 1$.

3. DOUBLE GEROCH GROUP STRUCTURE

In order to obtain a Geroch group structure, we have to use the dual formula (Zhong, 1990*b*)

$$d(J){:}\quad \mathscr{E}(J) = F(J) + J \cdot \Omega(J) \to D(J) = \hat{F}(\mathring{J}) + \mathring{J} \cdot \hat{\Omega}(\mathring{J}) \tag{21}$$

$$\hat{F}(\mathring{J}) = T(F(J))$$

$$\partial_\rho \hat{\Omega}(\mathring{J}) = \frac{J^2 \rho}{F^2(J)} \partial_z \Omega(J), \qquad \partial_z \hat{\Omega}(\mathring{J}) = -\frac{J^2 \rho}{F^2(J)} \partial_\rho \Omega(J)$$

where the symbol "∘" denotes the operation of substitution about the imaginary units, i.e.,

$$\mathring{i} = \varepsilon, \qquad \mathring{\varepsilon} = i \tag{22}$$

and the transformation T is

$$T\colon \ f \to T_{(f)} = \frac{\rho}{f} \tag{23}$$

$D(J)$ is a double solution of equation (1), too.

As for M and N, the dual formulas are

$$d(J)\colon \quad M_i(J)\begin{cases} M_i(i) \\ M_i(\varepsilon) \end{cases} \longrightarrow \tilde{M}_i(\mathring{J})\begin{cases} \tilde{M}_i(\varepsilon) \\ \tilde{M}_i(i) \end{cases} \tag{24}$$

$$N_i(J)\begin{cases} N_i(i) \\ N_i(\varepsilon) \end{cases} \longrightarrow \tilde{N}_i(\mathring{J})\begin{cases} \tilde{N}_i(\varepsilon) \\ \tilde{N}_i(i) \end{cases} \tag{25}$$

where

$$\begin{aligned} \tilde{M}_1(\mathring{J}) &= \frac{\partial_1 D(J)}{2\hat{F}(\mathring{J})}, \qquad \tilde{M}_2(\mathring{J}) = \frac{\partial_1 \bar{D}(J)}{2\hat{F}(\mathring{J})}, \qquad \tilde{M}_3(\mathring{J}) = \frac{1}{2\rho} \\ \tilde{N}_1(\mathring{J}) &= \frac{\partial_2 \bar{D}(J)}{2\hat{F}(\mathring{J})}, \qquad \tilde{N}_2(\mathring{J}) = \frac{\partial_2 D(J)}{2\hat{F}(\mathring{J})}, \qquad \tilde{N}_3(\mathring{J}) = \frac{1}{2\rho} \end{aligned} \tag{26}$$

in which $\tilde{M}_i$ and $\tilde{N}_i$ are still the double solutions of (9). It is worthwhile to note that in the ASVF case, from the gravitational field point of view, the gravitational solutions corresponding to $\tilde{M}_i$ and $\tilde{N}_i$ obtained from dual transformation $d(J)$ are equivalent to those corresponding to M_i and N_i. As a result, this mapping does not generate new solutions. Now let $\Sigma(M(J), N(J))$ denote the set of all nonpure imaginary solutions; then we obtain the following relation:

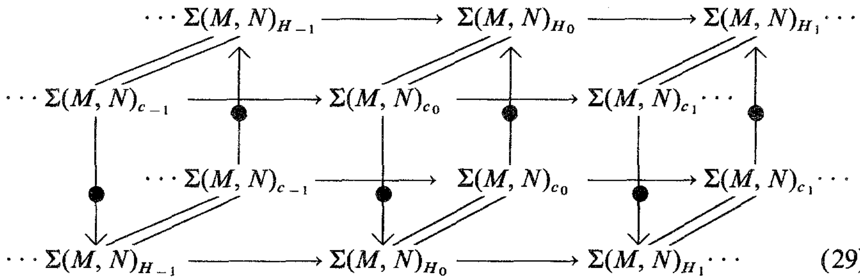
$$\begin{array}{ccc} \Sigma(M,N)_c & \overset{d_c}{=\!=\!=} & \Sigma(M,N)_H \\ {\scriptstyle j}\big\downarrow & & \big\uparrow{\scriptstyle j} \\ \Sigma(M,N)_H & \overset{d_H}{=\!=\!=} & \Sigma(M,N)_c \end{array} \tag{27}$$

where j is the operation of substituting i for ε; for $d(J)$ we have the relation

$$j \circ d_H \circ j = d_c, \qquad d_c \circ d_H = I, \qquad d_H \circ d_c = I \tag{28}$$

$d(J)$ is a one-to-one invertible mapping from $\Sigma(M, N)_J$ onto $\Sigma(M, N)_{\mathring{J}}$.

Now we can realize the double complex Geroch group in the ASVF case. Using (12) and (27), we obtain the following infinite 4-prismatic chain:



(29)

where the single-line arrow denotes the mapping I, the double line denotes the dual mapping $d(J)$, and the arrow with a dot denotes the mapping j. However, in this paper what is different from Zhong (1990*b*) is that the group transformations in the edges of the infinite 4-prismatic chain are more general.

The key to the realization of the Geroch group lies in the noncommutivity between the two transformations,

$$\begin{array}{ccc} \Sigma(M,N)_c & \xrightarrow{I_c} & \Sigma(M,N)_c \\ {\scriptstyle d_c}\Downarrow & & \Uparrow{\scriptstyle d_H} \\ \Sigma(M,N)_H & \xrightarrow{I_H} & \Sigma(M,N)_H \end{array} \tag{30}$$

$$d_H \circ I_H \circ d_c \neq I_c$$

Using this noncommutivity, we can realize the Geroch group along broken lines as

$$\longrightarrow (M,N)_{c_0} \longrightarrow (M,N)_{c_1} \Longrightarrow (\tilde{M},\tilde{N})_{H_1} -\cdot\rightarrow (\tilde{M},\tilde{N})_{c_1} \longrightarrow \tag{31}$$

Let us define

$$\begin{aligned} &\gamma_1(I;J) = I(J), \qquad \gamma_2(I;J) = d^{-1}(J)\circ I(J)\circ d(J) \\ &\gamma_3(I;J) = *, \qquad \gamma_4(I;J) = j(J)\circ d(J) \end{aligned} \tag{32}$$

where

$$j(J) = \begin{cases} j, & J = i \\ j^{-1}, & J = \varepsilon \end{cases} \tag{33}$$

We have the following results:

$$\begin{aligned} &\gamma_l(I;J)\circ\gamma_l(I';J) = \gamma_l(II';J) \\ &\gamma_l(I^{-1};J) = \gamma_l^{-1}(I;J); \qquad \gamma_l\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; J\right) = I \\ &(l = 1, 2) \end{aligned} \tag{34}$$

Let $G_1(\Sigma(M, N); J)$ denote the set of all the transformations with the following form:

$$\gamma_{r_m}(I_m; J) \circ \gamma_{r_{m-1}}(I_{m-1}; J) \circ \cdots \circ \gamma_{r_1}(I_1; J) \tag{35}$$

where the indexes $r_1, \ldots, r_m$ can be arbitrarily chosen (may have iteration) from the set $\{1, 2, 3, 4\}$. $G_1(\Sigma(M, N); J)$ is an infinite-dimensional group, and it is a double complex realization of the Geroch group. In addition, let $G_0(\Sigma(M, N), J)$ denote the subset while $r_1, r_2, \ldots, r_m \neq 3, 4$; then $G_0(\Sigma(M, N); J)$ is a simpler double complex realization of the Geroch group.

The above double complex realization is more general than in Zhong (1990*b*).

4. DOUBLE BÄCKLUND TRANSFORMATION AND GEROCH GROUP OF CSVF

Corresponding to the metric (5), the double complex equations for CSVF are

$$\frac{1-J^2}{2} M_{i,1} + \frac{1+J^2}{2} M_{i,2} = \frac{1-J^2}{2} (D_i^{kl} M_k M_l + E_i^{kl} M_k N_l) + \frac{1+J^2}{2} C_i^{kl} M_k N_l$$

$$\frac{1-J^2}{2} N_{i,2} + \frac{1+J^2}{2} N_{i,1} = \frac{1-J^2}{2} (D_i^{kl} N_k N_l + E_i^{kl} N_k M_l) + \frac{1+J^2}{2} C_i^{kl} N_k M_l$$

$$(i = 1, 2, 3) \tag{36}$$

If we adopt

$$\begin{aligned} M_1(J) &= \frac{\mathscr{C}_{,1}(J)}{2G(J)}, \qquad M_2(J) = \frac{\bar{\mathscr{C}}_{,1}(J)}{2G(J)}, \qquad M_3 = \frac{1}{2\rho} \\ N_1(J) &= \frac{\bar{\mathscr{C}}_{,2}(J)}{2G(J)}, \qquad N_2(J) = \frac{\mathscr{C}_{,2}(J)}{2G(J)}, \qquad N_3 = \frac{1}{2\rho} \end{aligned} \tag{37}$$

and the nonvanishing coefficients as in (11), in this case (36) will return to (6).

The mapping I given by (12) can be also applied to the CSVF case. Let $\alpha(J)$ and $\gamma(J)$ satisfy the Riccati equations (13) and (14); then through the mapping I the new solutions M and N can be obtained from known solution $\mathring{M}$ and $\mathring{N}$ in the same way.

The dual relation in the CSVF case is as follows:

$$\begin{aligned} \tilde{\partial}(J)\colon\ & \mathscr{C}(J) = G(J) + J \cdot M(J) \longrightarrow B(J) = \hat{G}(J) + J \cdot \hat{M}(J) \\ & \hat{G}(J) = T(G(J)) \\ & \partial_t \hat{M}(J) = \frac{t}{G^2(J)} \partial_z M(J), \qquad \partial_z \hat{M}(J) = \frac{t}{G^2(J)} \partial_t M(J) \end{aligned} \tag{38}$$

As for $M_i(J)$ and $N_i(J)$, the operation $\tilde{d}(J)$ is

$$\begin{aligned}\tilde{d}(J)\colon\quad M_i(J)\begin{cases}M_i(i)\\ M_i(\varepsilon)\end{cases} &\longrightarrow \tilde{M}_i(J)\begin{cases}\tilde{M}_i(i)\\ \tilde{M}_i(\varepsilon)\end{cases}\\ N_i(J)\begin{cases}N_i(i)\\ N_i(\varepsilon)\end{cases} &\longrightarrow \tilde{N}_i(J)\begin{cases}\tilde{N}_i(i)\\ \tilde{N}_i(\varepsilon)\end{cases}\end{aligned} \tag{39}$$

where

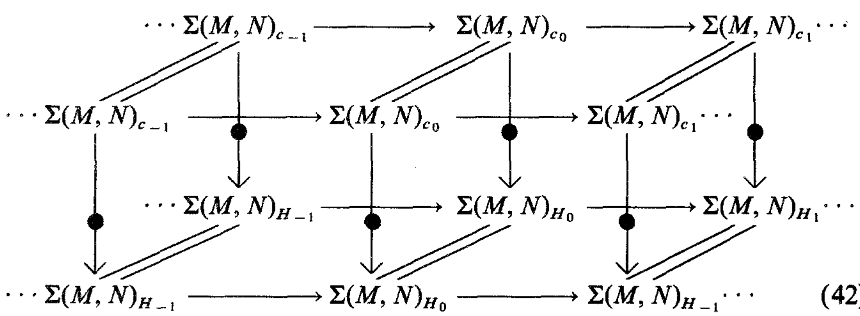
$$\begin{aligned}\tilde{M}_1(J) &= \frac{\partial_1 B(J)}{2\hat{G}(J)}, &\quad \tilde{M}_2(J) &= \frac{\partial_1 \bar{B}(J)}{2\hat{G}(J)}, &\quad \tilde{M}_3(J) &= \frac{1}{2\rho}\\ \tilde{N}_1(J) &= \frac{\partial_2 \bar{B}(J)}{2\hat{G}(J)}, &\quad \tilde{N}_2(J) &= \frac{\partial_2 \bar{B}(J)}{2\hat{G}(J)}, &\quad \tilde{N}_3(J) &= \frac{1}{2\rho}\end{aligned} \tag{40}$$

The $\tilde{M}_i(J)$ and $\tilde{N}_i(J)$ obtained in this way are still the solutions of (36). It is worth pointing out that in the CSVF case, when $J = i$, $\tilde{M}_i(J)$, $\tilde{N}_i(J)$ and $M_i(J)$, $N_i(J)$ correspond generally to different gravitational field solutions, respectively, i.e., the mapping $\tilde{d}(J = i)$ can generate new solutions, which is completely different from the ASVF case.

In the same way as before, let $\Sigma(M(J), N(J))$ denote the set consisting of all nonpure imaginary solutions; the following relation holds:

$$\begin{array}{ccc}\Sigma(M, N)_c & \overset{\tilde{d}_c}{=\!=\!=} & \Sigma(M, N)_c\\ \Big\downarrow j & & \Big\downarrow j\\ \Sigma(M, N)_H & \overset{\tilde{d}_H}{=\!=\!=} & \Sigma(M, N)_H\end{array} \tag{41}$$

Consequently, we present the following infinite 4-prismatic chain similar to that in the ASVF case:



(42)

The Geroch group is realized by using the following noncommutivity between two transformations:

$$
\begin{array}{ccc}
\Sigma(M,N)_c & \xrightarrow{I_c} & \Sigma(M,N)_c \\
\Big\Downarrow \tilde{d}_c & & \Big\Downarrow \tilde{d}_c \\
\Sigma(M,N)_c & \xrightarrow{I_c} & \Sigma(M,N)_c
\end{array}
\qquad
\begin{array}{ccc}
\Sigma(M,N)_H & \xrightarrow{I_H} & \Sigma(M,N)_H \\
\Big\Downarrow \tilde{d}_H & & \Big\Downarrow \tilde{d}_H \\
\Sigma(M,N)_H & \xrightarrow{I_H} & \Sigma(M,N)_H
\end{array}
\qquad (43)
$$

$$
\tilde{d}_c \circ I_c \neq I_c \circ \tilde{d}_c \qquad\qquad \tilde{d}_H \circ I_H \neq I_H \circ \tilde{d}_H
$$

In CSVF, because $M(i)$, $N(i)$ and $\tilde{M}(i)$, $\tilde{N}(i)$ correspond, respectively, to the different solutions, we must consider the pair $((M(J), N(J)),$ $(\tilde{M}(J), \tilde{N}(J)))$ of dual solutions, which must be still be a pair of dual solutions under a transformation in the realization to be found. We construct γ_1 and γ_2 along the following lines in the upper and the lower faces of the prism:

$$
\begin{array}{ccc}
\tilde{M},\tilde{N}\ \bullet & & \bullet\ \tilde{M}',\tilde{N}' \\
\Big\Uparrow & \gamma_1 & \Big\Uparrow \\
M,N\ \bullet & \longrightarrow & \bullet\ M',N'
\end{array}
\qquad
\begin{array}{ccc}
\tilde{M},\tilde{N}\ \bullet & \longrightarrow & \bullet\ \tilde{M}',\tilde{N}' \\
\Big\Downarrow & \gamma_2 & \Big\Downarrow \\
M,N\ \bullet & & \bullet\ M',N'
\end{array}
\qquad (44)
$$

Furthermore, let us define

$$
\begin{gathered}
\gamma_1(I;J):\quad [(M,N),(\tilde{M},\tilde{N})] \to [I(J)(M,N), \tilde{d}(J) \circ I(J) \circ \tilde{d}^{-1}(\tilde{M},\tilde{N})] \\
\gamma_2(I;J):\quad [(M,N),(\tilde{M},\tilde{N})] \to [\tilde{d}^{-1}(J) \circ I(J) \circ \tilde{d}(J)(M,N), I(J)(\tilde{M},\tilde{N})] \\
\gamma_3(I;J) = *, \qquad \gamma_4(I;J):\quad [(M,N),(\tilde{M},\tilde{N})] \to [(\tilde{M},\tilde{N}),(M,N)] \qquad (45)
\end{gathered}
$$

Equations (34) still hold here. Therefore, we define $G_0(\Sigma(M,N)_c; J)$ and $G_1(\Sigma(M,N)_c; J)$ in the same way as in (35); both are double complex realizations of the Geroch group in CSVF.

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